

Partially integrable nonlinear equations with one higher symmetry

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys. A: Math. Gen. 38 L337

(<http://iopscience.iop.org/0305-4470/38/20/L02>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.66

The article was downloaded on 02/06/2010 at 20:13

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

Partially integrable nonlinear equations with one higher symmetry

A V Mikhailov^{2,3}, V S Novikov^{1,3} and J P Wang¹¹ Institute of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NF, UK² Department of Applied Mathematics, University of Leeds, Woodhouse Lane, Leeds LS2 9JT, UK³ L D Landau Institute for Theoretical Physics, Kosygina St. 2, Moscow 117332, Russia

E-mail: sashamik@maths.leeds.ac.uk, V.Novikov@kent.ac.uk and J.Wang@kent.ac.uk

Received 12 April 2005

Published 3 May 2005

Online at stacks.iop.org/JPhysA/38/L337**Abstract**

In this letter, we present a family of second order in time nonlinear partial differential equations, which have only one higher symmetry. These equations are not integrable, but have a solution depending on one arbitrary function.

PACS numbers: 02.30.Jr, 02.30.Ik

1. Introduction

For a long time there has been a general belief, ‘if a partial differential equation or a system of differential equations has one nontrivial symmetry, then it has infinitely many’ [1]. Indeed, this statement is true in the case of evolutionary equations, in which the right-hand side is a homogeneous differential polynomial [2]. However, in 1991 Bakirov proposed an example putting this conjecture in doubt [3]. He found that the system

$$\begin{cases} u_t = u_{xxxx} + v^2 \\ v_t = \frac{1}{5}v_{xxxx} \end{cases} \quad (1)$$

has a symmetry of order 6 and he also verified using a computer algebra software that it does not have other symmetries up to the order 53. Recently, using p -adic analysis, it has been rigorously proven that system (1) has only one higher symmetry and therefore the above conjecture is not valid [4]. Furthermore, there exist infinitely many systems of the same type with only one high symmetry. Even a modified conjecture, stating that the existence of n higher symmetries for an n -component system of evolutionary equations implies infinitely many [5], is not valid either—there exists a system, similar to (1) of order 7, which has only two higher symmetries [6].

The Cauchy problem for systems of the form (1) is always solvable no matter how many symmetries the system has. One can solve the second equation and substitute the solution into the first one. The equation obtained becomes a linear inhomogeneous equation, which can be solved using standard techniques.

In this letter, we present the following family of partial differential equations:

$$u_{tt} = u_{n,t} - \frac{u_t}{u}(u_n - u_t) + \frac{(u_n - u_t)^2}{u} - u \partial_x^n \left(\frac{u_n - u_t}{u} \right), \quad n \geq 2. \quad (2)$$

Here and further below we adopt the notation $u_k \equiv \partial_x^k(u)$. The result of our study can be formulated as the following:

Theorem 1. For any $n \geq 2$ equation (2) possesses a local higher symmetry

$$u_\tau = \frac{1}{u}(u_n - u_t). \quad (3)$$

This symmetry is the only local higher symmetry of equation (2).

Equation (2) is not integrable by any known methods. Nevertheless, there is a family of exact solutions depending on one arbitrary function.

2. Proof of the theorem and discussion

There are many equivalent definitions of infinitesimal local higher symmetries for partial differential equations (see, for instance, [7]). In this letter we adopt the following definition. A partial differential equation $u_\tau = G(u_m, u_{m-1}, \dots, u, u_{s,t}, u_{s-1,t}, \dots, u_t)$ generates a local symmetry of equation (2) if it is compatible with equation (2) [8]. A symmetry is called a higher symmetry of equation (2) if $m > 1$ or $s > 1$.

Surprisingly enough, equation (2) and its symmetry (3) can be rewritten in a very compact form

$$z_t = z_n - z^2, \quad (4)$$

$$u_\tau = z, \quad (5)$$

where

$$z = \frac{1}{u}(u_n - u_t). \quad (6)$$

For any $n \geq 2$ equation (4) does not possess higher symmetries and is not integrable by the inverse scattering method or solvable by any other methods. Its trivial solution $z = 0$ corresponds to a stationary point of symmetry (5) and provides a nontrivial family of solutions to equation (2). Indeed, $z = 0$ implies that

$$u_t = u_n.$$

The general solution of this linear equation depends on one arbitrary function and it can be found by standard methods.

Proof of the theorem. Let us rewrite equation (2) as a system of two evolutionary equations on variables u and $v = u_t$ as follows:

$$\begin{aligned} u_t &= v, \\ v_t &= v_n - \frac{v}{u}(u_n - v) + \frac{(u_n - v)^2}{u} - u \partial_x^n \left(\frac{u_n - v}{u} \right) \equiv H[u, v]. \end{aligned} \quad (7)$$

Equation (3) is a symmetry of (2) if and only if system (7) is compatible with

$$\begin{aligned} u_\tau &= \frac{1}{u}(u_n - v), \\ v_\tau &= D_t \left(\frac{1}{u}(u_n - v) \right) = \partial_x^n \left(\frac{u_n - v}{u} \right) - \frac{(u_n - v)^2}{u^2}, \end{aligned} \tag{8}$$

where D_t stands for the operator of total differentiation with respect to t according to (7). Calculating the cross derivatives and expressing the results in terms of variable z (6) we obtain

$$\begin{aligned} u_{t\tau} &= v_\tau = z_n - z^2, \\ u_{\tau t} &= z_t = z_n - z^2 \end{aligned}$$

and hence $u_{t\tau} = u_{\tau t}$. Analogously,

$$\begin{aligned} v_{t\tau} &= \partial_x^n(z_n - z^2) - 2z(z_n - z^2) - u\partial_\tau(z_n - z^2) - vz_\tau, \\ v_{\tau t} &= \partial_x^n(z_n - z^2) - 2z(z_n - z^2). \end{aligned}$$

It is easy to verify that $z_\tau = 0$ and therefore $v_{t\tau} = v_{\tau t}$ and hence (3) is the symmetry of equation (2).

Let us prove that symmetry (3) is the only local higher symmetry of equation (2). In our proof, we shall use elements of the perturbative symmetry approach in the symbolic representation [2, 9].

Function $H[u, v]$ on the right-hand side of the system (7) can be rewritten in the form

$$H[u, v] = -u_{2n} + 2u_{n,t} + \sum_{i=1}^n f_i(u) P_i[u, v], \tag{9}$$

where $f_i(u) = \frac{1}{u^i}$ and $P_i[u, v]$ are polynomials in v and derivatives of u and v with respect to x with constant coefficients, such that

$$P_i[\lambda u, \lambda v] = \lambda^{i+1} P_i[u, v], \quad \lambda \in C. \tag{10}$$

Equation (2) is a homogeneous equation, therefore polynomials P_i are homogeneous polynomials of the weight $2n$, i.e. $W(P_i) = 2n P_i$, where

$$W = \sum_{k=1}^{\infty} k u_k \frac{\partial}{\partial u_k} + \sum_{k=0}^{\infty} (k+n) v_k \frac{\partial}{\partial v_k}.$$

Function $H[u, v]$ can be treated as a differential polynomial in v and derivatives of u, v with coefficients being functions of u only. The symbolic representation of such polynomials has been defined and studied in [10].

To prove the second statement of the theorem we will need only a few first terms of the function $H[u, v]$ (9):

$$\begin{aligned} H[u, v] &= -u_{2n} + 2v_n + f_1(u) (\partial_x^n(uu_n - uv) - uu_{2n} + u_n^2 + uv_n - 3u_nv + 2v^2) + R[u, v], \\ \text{where } R[u, v] &= \sum_{i \geq 2} f_i(u) P_i[u, v]. \end{aligned}$$

Then in the symbolic representation system (7) can be rewritten as

$$\begin{aligned} u_t &= \hat{v}, \\ v_t &= -\hat{u}k_1^{2n} + 2\hat{v}q_1^n + f_1(u) \circ [\hat{u}^2 a_1(k_1, k_2) + \hat{u}\hat{v} a_2(k_1, q_1) + 2\hat{v}^2] + \hat{R}[u, v], \end{aligned} \tag{11}$$

where

$$\begin{aligned} a_1(k_1, k_2) &= \frac{1}{2} [(k_1 + k_2)^n (k_1^n + k_2^n) - (k_1^n - k_2^n)^2], \\ a_2(k_1, q_1) &= q_1^n - (k_1 + q_1)^n - 3k_1^n, \end{aligned}$$

and $\hat{R}[u, v]$ stands for the symbolic representation of $R[u, v]$.

It is easy to verify that the most general form of a higher symmetry ($m \geq 2$) of system (7) in the symbolic representation is

$$u_\tau = g_1(u) \circ \hat{u} k_1^m + g_2(u) \circ \hat{v} q_1^{m-n} + \hat{u}^2 A_1(k_1, k_2; u) \\ + \hat{u} \hat{v} A_2(k_1, q_1; u) + \hat{v}^2 A_3(q_1, q_2; u) + \hat{S}[u, v] \equiv G, \quad (12)$$

$$v_\tau = D_t(G).$$

Here $A_i(x, y; u)$ are polynomials in x, y with coefficients, depending on u . Polynomials $A_i(x, y; u)$ satisfy the condition $A_i(0, y; u) = 0$, $i = 1, 2$. The remainder $\hat{S}[u, v]$ stands for the terms of higher nonlinearity in v and x derivatives of u, v . \square

Lemma 1. *If (12) is a symmetry of equation (11), then $g_1(u)$ and $g_2(u)$ are not equal to zero simultaneously.*

Sketch of the proof. Let us assume that symmetry (12) does not have linear terms (in \hat{u}, \hat{v}) and starts with the terms of order $s > 1$:

$$u_\tau = \sum_{p=0}^s A_p(k_1, k_2, \dots, k_p, q_1, q_2, \dots, q_{s-p}; u) \hat{u}^p \hat{v}^{s-p} + \text{higher order terms.}$$

It follows from the compatibility conditions of (11) and (12) that the coefficients $A_p(k_1, k_2, \dots, k_p, q_1, q_2, \dots, q_{s-p}; u)$ satisfy a linear system of homogeneous algebraic equations. This system has only a trivial solution $A_p = 0$, $p = 0, \dots, s$ since the determinant of the corresponding matrix is not identically vanishing.

The case $m = n$ corresponds to symmetry (3). In this case $A_i(x, y; u) = 0$ and $g_1(u) = -g_2(u) = \frac{1}{u}$.

Let $m \neq n$. From the compatibility conditions of (11) and (12) it follows that

$$A_3(q_1, q_2; u) = \frac{1}{2} f_1(u) g_2(u) \frac{q_1^{m-n} + q_2^{m-n} - 2(q_1 + q_2)^{m-n}}{q_1^n + q_2^n - (q_1 + q_2)^n}.$$

Polynomial $q_1^n + q_2^n - (q_1 + q_2)^n$ does not divide $q_1^{m-n} + q_2^{m-n} - 2(q_1 + q_2)^{m-n}$ if $m \neq n$ and therefore $A_3(q_1, q_2; u)$ does not represent a symbol of a differential polynomial with u -dependent coefficients. It implies that $g_2(u) = 0$.

Having $g_2(u) = 0$ we find (from the compatibility conditions) that

$$A_2(k_1, q_1; u) = f_1(u) g_1(u) \frac{k_1^m + q_1^m - (k_1 + q_1)^m}{k_1^n + q_1^n - (k_1 + q_1)^n}.$$

For $s > 1$ the polynomial $K_s(x, y) = x^s + y^s - (x + y)^s$ can be factorized as

$$K_s(x, y) = xy T_s(x, y), \quad T_s(0, y) \neq 0. \quad (13)$$

Hence

$$A_2(k_1, q_1; u) = f_1(u) g_1(u) \frac{T_m(k_1, q_1)}{T_n(k_1, q_1)}. \quad (14)$$

It follows from (13) that $A_2(0, q_1; u) = 0$ only if $g_1(u) = 0$. It follows from the lemma that a symmetry without a linear part does not exist. \square

Acknowledgments

VSN is funded by a Royal Society NATO/Chevening fellowship on the project Symmetries and Coherent Structures in Non-Evolutionary Partial Differential Equations. AVM and VSN thank the RFBR grant no 02-01-00431 for partial support. AVM is particularly grateful for the invitation to the Visiting Thomas Stieltjes Professorial Fellowship, during which some of this work has been carried out.

References

- [1] Fokas A S 1980 A symmetry approach to exactly solvable evolution equations *J. Math. Phys.* **21** 1318–25
- [2] Sanders J A and Wang J P 1998 On the integrability of homogeneous scalar evolution equations *J. Diff. Eqns* **147** 410–34
- [3] Bakirov I M 1991 On the symmetries of some system of evolution equations *Technical Report* Akad. Nauk SSSR Ural. Otdel. Bashkir. Nauchn. Tsentr, Ufa
- [4] Beukers F, Sanders J A and Wang J P 1998 One symmetry does not imply integrability *J. Diff. Eqns* **146** 251–60
- [5] Fokas A S 1987 Symmetries and integrability *Stud. Appl. Math.* **77** 253–99
- [6] van der Kamp P H and Sanders J A 2002 Almost integrable evolution equations *Sel. Math.* **8** 705–19
- [7] Mikhaïlov A V, Shabat A B and Sokolov V V 1991 The symmetry approach to classification of integrable equations *What is integrability? (Nonlinear Dynamics)* (Berlin: Springer) pp 115–84
- [8] Hernández Heredero R, Shabat A B and Sokolov V V 2003 A new class of linearizable equations *Preprint* nlin SI/0301001 (*Phys. Lett. A* at press)
- [9] Mikhailov A V and Novikov V S 2002 Perturbative symmetry approach *J. Phys. A: Math. Gen.* **35** 4775–90
- [10] Sanders J A and Wang J P 2000 On the integrability of non-polynomial scalar evolution equations *J. Diff. Eqns* **166** 132–50